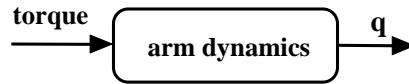


## What is dynamics?

A description of how forces acting on a system result in motion of that system.



**Example:** A ball of mass  $m$  is held 20 m off the ground. The force acting on the ball is the force of gravity:  $f = -mg$  where  $g = 9.8 \text{ m/s}^2$ . If we drop the ball, its dynamics are describe by:

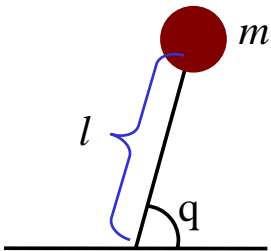
$$x(0) = 20 \text{ m}$$

$$\dot{x}(0) = 0 \text{ m/s}$$

$$f = -mg$$

$$\ddot{x}(t) = \frac{f}{m} = -g = -9.8 \rightarrow x(t) = 20 - 4.9t^2$$

**Example:** Dynamics of a single joint system with mass  $m$  and length  $l$ .



$$\ddot{q} = \frac{1}{ml^2} (\tau - mgl \cos q)$$

# Path of motion of a system is one that minimizes an energy cost

Imagine a point mass that is at position  $x_1$  at time  $t_1$  and ends up at position  $x_2$  at  $t_2$ , for example: a ball falling from a height. The trajectory that it follows to get to  $x_2$  is only one of an infinite number of pathways that it could have followed. But the point mass will always follow that same trajectory  $x(t)$ , given the same initial conditions. What is so special about the trajectory  $x(t)$  that it actually does follow?

The trajectory  $x(t)$  minimizes the following cost function:

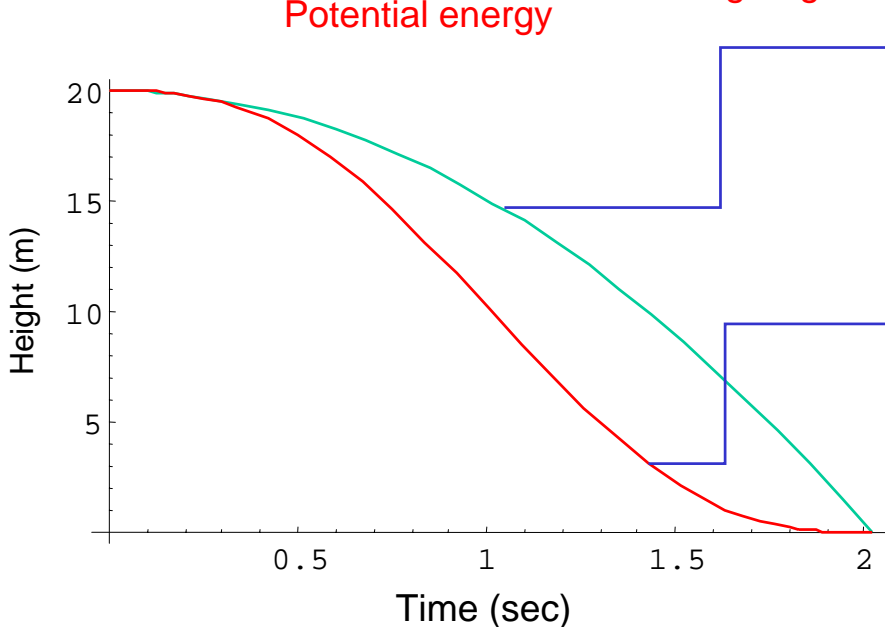
$$H(x(t)) = \int_{t_1}^{t_2} (KE - PE) dt = \int_{t_1}^{t_2} L(x, \dot{x}, t) dt$$

Kinetic energy

Potential energy

Lagrangian

For a point mass:  $\begin{cases} KE = \frac{1}{2} m \dot{x}^2 \\ PE = mgx \end{cases}$



$$\ddot{x}(t) = -9.8 \rightarrow x(t) = 20 - 4.9t^2$$

$$H(x(t)) = \int_0^{2.02} \left( \frac{1}{2} m \dot{x}(t)^2 - mgx(t) \right) dt = -264$$

$$x(t) = 20 + 20(-1.21t^3 + 0.9t^4 - 0.178t^5)$$

$$H(x(t)) = \int_0^{2.02} \left( \frac{1}{2} m \dot{x}(t)^2 - mgx(t) \right) dt = -113$$

## Solving the functional for a point mass

$$H(x(t)) = \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} (KE - PE) dt = \int_{t_1}^{t_2} \frac{1}{2} m \dot{x}^2 - mgx dt$$

$$x(t) \rightarrow x(t) + e\eta(t)$$

$$H(x + e\eta) = \int_{t_1}^{t_2} \frac{1}{2} m (\dot{x} + e\dot{\eta})^2 dt - \int_{t_1}^{t_2} mg(x + e\eta) dt$$

$$H(x + e\eta) = \int_{t_1}^{t_2} \frac{1}{2} m (\dot{x} + e\dot{\eta})^2 dt - \int_{t_1}^{t_2} mg(x + e\eta) dt$$

$$\left. \frac{dH}{de} \right|_{e=0} = \int_{t_1}^{t_2} m(\dot{x} + e\dot{\eta})\dot{\eta} dt - \int_{t_1}^{t_2} mg\eta dt$$

$$\left. \frac{dH}{de} \right|_{e=0} = \int_{t_1}^{t_2} m\dot{x}\dot{\eta} dt - \int_{t_1}^{t_2} mg\eta dt$$

$$u = \dot{x} \quad dv = \dot{\eta} dt \quad du = \ddot{x} dt \quad v = \eta$$

$$\int_{t_1}^{t_2} m\dot{x}\dot{\eta} dt = m\dot{x}\eta \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} m\eta\ddot{x} dt = - \int_{t_1}^{t_2} m\ddot{x}\eta dt$$

$$\left. \frac{dH}{de} \right|_{e=0} = - \int_{t_1}^{t_2} m\ddot{x}\eta dt - \int_{t_1}^{t_2} mg\eta dt = 0$$

$$-m\ddot{x} - mg = 0 \rightarrow -mg = m\ddot{x}$$

## General solution for the functional

$$H(x(t)) = \int_{t_1}^{t_2} (KE - PE) dt = \int_{t_1}^{t_2} \underbrace{L(x, \dot{x}, t)}_{\text{Lagrangian}} dt$$

$$\frac{d}{dt} \left( \frac{dL}{d\dot{x}} \right) - \frac{dL}{dx} = 0$$

The solution to the calculus of variation approach to minimize  $H(x)$

**Example:** dynamics of a point mass

$$\left. \begin{aligned} L &= \frac{1}{2} m \dot{x}^2 - mgx \\ \frac{dL}{d\dot{x}} &= m\dot{x} \quad \frac{d}{dt} \left( \frac{dL}{d\dot{x}} \right) = m\ddot{x} \quad \frac{dL}{dx} = -mg \end{aligned} \right\} m\ddot{x} + mg = 0 \rightarrow -mg = m\ddot{x}$$

If there are external forces (from motors, muscles) acting on the system:

$$\frac{d}{dt} \left( \frac{dL}{d\dot{x}} \right) - \frac{dL}{dx} = F$$

The primary problem in dynamics is to find an expression for the kinetic energy of the system.